Equilibriums and Mixing Time in Open Quantum Systems

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The mixing time of an open quantum system is an important question. Previous literature often relies on estimating the spectral gap of the Lindbladian generator. A recent interesting work [2] proposed a theoretical framework of obtaining the mixing time, without relying on the estimation of the spectral gap, based on several assumptions. Here I reviewed some related mathematical concepts in functional analysis and discussed the high-level ideas in [2].

1 A Brief Review of Dual Space

In this section the materials are mainly re-organized from wikipedia [1, 3, 5].

Definition 1 (Dual Space)**.** *Given any vector space V over a field F, the dual space is defined V^* as the set of all linear maps $\phi : V \to F$. The dual space V^* itself becomes a vector space over F:*

$$\begin{aligned}(\phi+\psi)(x) &= \phi(x)+\psi(x),\\ (a\phi)(x) &= a\phi(x).\end{aligned} \tag{1}$$

Here $x \in V$, ϕ, ψ are in V^* and $a \in F$.

Definition 2 (Algebraic Adjoint)**.** *Let X^* denote the dual space of a vector space X, and Y^* is the dual space of a vector space Y. If $u : X \to Y$ is a linear map, then its adjoint map $u^* : Y^* \to X^*$ is defined as $u^* f = fu$, where $f \in Y^*$. This can also be characterized as*

$$\langle u^* f, x\rangle = \langle f, ux\rangle, \tag{2}$$

where $x \in X$ and $ux \in Y$.

For some matrix $A \in \mathbb{C}^{m\times n}$, $x \in \mathbb{C}^n, y \in \mathbb{C}^m$, take the usual inner product in the complex domain, we have

$$\langle Ax, y\rangle = (Ax)^\dagger y = x^\dagger A^\dagger y = \langle x, A^\dagger y\rangle. \tag{3}$$

Thus it is easy to see that for a finite-dimensional linear map described as a matrix A, its adjoint is its conjugate transpose $A^\dagger$.

Now let us consider a Lindbladian master equation:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\rho = \mathcal{L}^*\rho,\\ &\mathcal{L}^* = -\imath[H,(\cdot)] + \sum_j [V_j(\cdot), V_j^\dagger] + [V_j, (\cdot)V_j^\dagger].\end{aligned} \tag{4}$$

Here ρ is some density operator in a vector space X. For some observable A, we can view A as a member in X^*, where

$$\langle A, \rho\rangle := A(\rho) = \mathrm{Tr}(A\rho). \tag{5}$$

Consider L^* as a linear map that maps from X to X, then its adjoint L maps an observable to another observable, defined as

$$\mathcal{L} = \imath[H, (\cdot)] + \sum_j [V_j^\dagger (\cdot), V_j] + [V_j^\dagger, (\cdot) V_j] = \mathcal{H} + \mathcal{D}, \tag{6}$$

which is simply the conjugate transpose of L^*. Thus the adjoint dynamics is

$$\frac{\mathrm{d}}{\mathrm{d}t} A = \mathcal{L} A. \tag{7}$$

Not let us re-consider Equation 5.

$$\langle e^{\mathcal{L}t} A, \rho \rangle = \langle A, e^{\mathcal{L}^* t} \rho \rangle. \tag{8}$$

This inspired us that if $e^{\mathcal{L}^* t}\rho$ goes to its equilibrium σ,

$$\lim_{t\to\infty} \langle e^{\mathcal{L}t} A, \rho \rangle = \lim_{t\to\infty} \langle A, e^{L^* t} \rho \rangle = \langle A, \sigma \rangle = \langle A, e^{\mathcal{L}^* \tau} \sigma \rangle = \langle e^{\mathcal{L}\tau} A, \sigma \rangle. \tag{9}$$

Again make τ goes to infinity, we can see the limit of $e^{\mathcal{L}t}A$ should be $\mathrm{Tr}(A\sigma)\mathbb{I}$, because of the arbitrary choice of ρ. We can foresee that the speed of $e^{\mathcal{L}t}A$ goes to $\mathrm{Tr}(A\sigma)\mathbb{I}$ could be used to quantify the speed of $e^{\mathcal{L}^* t}\rho$ goes to σ.

Remark 3 ([4])**.** *For a CPTP map ϕ, its adjoint ϕ^* is unital, i.e. $\phi^*(\mathbb{I}) = \mathbb{I}$. It is worth noting that actually for any $k\mathbb{I}$, $\phi^*(k\mathbb{I}) = k\mathbb{I}$.*

2 High-level Overview of the Estimation

As I stated before, now we turn to quantify the speed of $e^{\mathcal{L}t}A$ goes to $\mathrm{Tr}(A\sigma)\mathbb{I}$. Given the GNS inner-product defined as

$$\langle A, B \rangle_{\mathrm{GNS}} := \mathrm{Tr}\left(\sigma A^\dagger B\right). \tag{10}$$

Note here in the original paper [2], they are stating an equilibrium $\mathbb{I}$.

So the distance between A and the equilibrium $\mathrm{Tr}(A\sigma)\mathbb{I}$ is

$$\mathcal{A} := A - \frac{\langle A, \mathrm{Tr}(A\sigma)\mathbb{I} \rangle_{\mathrm{GNS}}}{\langle \mathrm{Tr}(A\sigma)\mathbb{I}, \mathrm{Tr}(A\sigma)\mathbb{I} \rangle_{\mathrm{GNS}}} \mathrm{Tr}(A\sigma)\mathbb{I} = A - \mathrm{Tr}\left(A\sigma\right)\mathbb{I}. \tag{11}$$

Remark 4. *This can also be easily understood as the difference between the starting point and its equilibrium.*

An important property here is that $\mathrm{Tr}\left(\mathcal{A}\sigma\right) = 0$. Consider the set of Hermitian operators $\mathcal{S}_{\mathcal{A}} := \{\mathcal{A} : \mathrm{Tr}\left(\mathcal{A}\sigma\right) = 0\}$, we are hoping that $e^{\mathcal{L}t}$ damps any element in $\mathcal{S}_{\mathcal{A}}$. If this holds true,

$$\begin{aligned}
\left\| e^{\mathcal{L}^* t}\rho - \sigma \right\|_1 &= \sup_{-\mathbb{I} \le A \le \mathbb{I}} \mathrm{Tr}\left(A(e^{\mathcal{L}^* t}\rho - \sigma)\right) = \sup_{-\mathbb{I} \le A \le \mathbb{I}} \mathrm{Tr}\left(A e^{\mathcal{L}^* t}\rho - A\sigma\right) \\
&= \sup_{-\mathbb{I} \le A \le \mathbb{I}} \mathrm{Tr}\left(A e^{\mathcal{L}^* t}\rho\right) - \mathrm{Tr}\left(A\sigma\right) = \sup_{-\mathbb{I} \le A \le \mathbb{I}} \langle A, e^{\mathcal{L}^* t}\rho \rangle - \mathrm{Tr}\left(A\sigma\right) \\
&= \sup_{-\mathbb{I} \le A \le \mathbb{I}} \langle e^{\mathcal{L}t} A, \rho \rangle - \mathrm{Tr}\left(A\sigma\right) = \sup_{-\mathbb{I} \le A \le \mathbb{I}} \langle e^{\mathcal{L}t}(A - \mathrm{Tr}\left(A\sigma\right)\mathbb{I}), \rho \rangle + \mathrm{Tr}\left(A\sigma\right) - \mathrm{Tr}\left(A\sigma\right) \\
&= \sup_{-\mathbb{I} \le A \le \mathbb{I}} \langle e^{\mathcal{L}t}(A - \mathrm{Tr}\left(A\sigma\right)\mathbb{I}), \rho \rangle = \sup_{-\mathbb{I} \le A \le \mathbb{I}} \mathrm{Tr}\left(e^{\mathcal{L}t}(A - \mathrm{Tr}\left(A\sigma\right)\mathbb{I})\rho\right) \\
&= \sup_{-\mathbb{I} \le A \le \mathbb{I}} \mathrm{Tr}\left(\sigma\sigma^{-1}\rho e^{\mathcal{L}t}(A - \mathrm{Tr}\left(A\sigma\right)\mathbb{I})\right) \\
&= \langle \sigma^{-1}\rho, e^{\mathcal{L}t}(A - \mathrm{Tr}\left(A\sigma\right)\mathbb{I}) \rangle_{\mathrm{GNS}} \le \|\sigma^{-1}\rho\|_{\mathrm{GNS}} \left\| e^{\mathcal{L}t}(A - \mathrm{Tr}\left(A\sigma\right)\mathbb{I}) \right\|_{\mathrm{GNS}}.
\end{aligned} \tag{12}$$

In fact, it is possible to have the theorem below:

Theorem 5 (Main result in [2]). *Under certain conditions, there exists positive constants λ and C, such that for some Hermitian operator A satisfying $Tr(A\sigma) = 0$, we have*

$$\left\|e^{\mathcal{L}t} A\right\|_{GNS} \leq C e^{-\lambda t} \|A\|_{GNS}. \tag{13}$$

Now we need to design the conditions needed to damp the elements in $\mathcal{S}_{\mathcal{A}}$. Take $A \in \mathcal{S}_{\mathcal{A}}$, define $A(t) = e^{t\mathcal{L}} A = e^{t(\mathcal{D}+\mathcal{H})} A$. Condiser $L(A(t)) := \frac{1}{2} \|A(t)\|_{\mathrm{GNS}}^2$, and its derivative is $-D(A(t)) := \frac{\mathrm{d}}{\mathrm{d}t} L(A(t)) = \frac{1}{2} \langle (\mathcal{D} + \mathcal{H}) A(t), A(t) \rangle_{\mathrm{GNS}} + \frac{1}{2} \langle A(t), (\mathcal{D} + \mathcal{H}) A(t) \rangle_{\mathrm{GNS}} = \Re \langle \mathcal{D} A(t), A(t) \rangle_{\mathrm{GNS}}$.

We are hoping to show that $D(A(t)) \geq \kappa \|A(t)\|_{\mathrm{GNS}}$ for some positive κ, and this can be achieved by providing some assumptions on $\mathcal{H}$ and $\mathcal{D}$. The details can be found in [2].

Remark 6. *Actually one shuold use an equivalent norm of $L(A)$ other than the exact $L(A)$ itself to prove the desired properties.*

References

[1] *Dual space*. https://en.wikipedia.org/wiki/Dual_space.

[2] Di Fang, Jianfeng Lu, and Yu Tong. "Mixing time of open quantum systems via hypocoercivity". In: *arXiv preprint arXiv:2404.11503* (2024).

[3] *Functional*. https://en.wikipedia.org/wiki/Functional_(mathematics).

[4] *Quantum channel*. https://en.wikipedia.org/wiki/Quantum_channel.

[5] *Transpose of a linear map*. https://en.wikipedia.org/wiki/Transpose_of_a_linear_map.