# A Very First Step in Qubit Control for Superconducting Qubits

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As I delve deeper in quantum computing, the more I feel like about learning how a quantum computer really works. Introduced by Junyi Liu, I write this note based on his post and the Guide [2]. I start by briefly review some basics of quantum harmonic oscillator and reproduce the simplest results for qubit control, i.e. the implementation of single-qubit gates.

## 1 Quantum Harmonic Oscillator

### 1.1 The Position Operator and Momentum Operator

For a classical particle, we may interested in its position and momentum. For a quantum state $\Psi(x,t)$, we can only get the estimation from expectation. Note that it satisfies the Schrödinger equation, say

$$\begin{aligned}\frac{\partial \Psi}{\partial t} &= \frac{\imath\hbar}{2m}\Psi_{xx} - \frac{\imath}{\hbar}V\Psi,\\ \frac{\partial \Psi^*}{\partial t} &= -\frac{\imath\hbar}{2m}\Psi^*_{xx} + \frac{\imath}{\hbar}\Psi^* V.\end{aligned} \tag{1}$$

Following the reference [1], the expectation value of the position $x$ is

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 \mathrm{d}x = \int_{-\infty}^{\infty} \Psi^*[x]\Psi \mathrm{d}x. \tag{2}$$

To see how $\langle x\rangle$ changes,

$$\begin{aligned} m\frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle &= m\int_{-\infty}^{\infty} x\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(x,t)|^2\mathrm{d}x = m\int_{-\infty}^{\infty} x(\Psi^*\Psi_t + \Psi^*_t\Psi)\mathrm{d}x\\ &= m\int_{-\infty}^{\infty} x(\Psi^*(\frac{\imath\hbar}{2m}\Psi_{xx} - \frac{\imath}{\hbar}V\Psi) + (-\frac{\imath\hbar}{2m}\Psi^*_{xx} + \frac{\imath}{\hbar}\Psi^*V)\Psi)\mathrm{d}x\\ &= m\int_{-\infty}^{\infty} x\frac{\imath\hbar}{2m}(\Psi^*\Psi_{xx} - \Psi^*_{xx}\Psi)\mathrm{d}x = \frac{\imath\hbar}{2}\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}(\Psi^*\Psi_x - \Psi^*_x\Psi)\mathrm{d}x\\ &= -\frac{\imath\hbar}{2}\int_{-\infty}^{\infty}\Psi^*\Psi_x - \Psi^*_x\Psi\mathrm{d}x = -\imath\hbar\int_{-\infty}^{\infty}\Psi^*\Psi_x\mathrm{d}x = \int_{-\infty}^{\infty}\Psi^*[-\imath\hbar\frac{\partial}{\partial x}]\Psi\mathrm{d}x.\end{aligned} \tag{3}$$

With the computation above, we conclude two operators $\hat{x}$ and $\hat{p}$, where

$$\begin{aligned}\hat{x}\Psi(s,t) &:= s\Psi(s,t),\\ \hat{p}\Psi(x,t) &:= -\imath\hbar\frac{\partial}{\partial x}\Psi(x,t).\end{aligned} \tag{4}$$

These two operators satisfy the canonical commutation relation, say

$$[\hat{x},\hat{p}] = \imath\hbar, \tag{5}$$

which is easy to verify.

## 1.2 The Harmonic Oscillator

A perfect harmonic oscillator is a mass $m$ attached to a spring of force constant $k$. The classical Hamiltonian is written by

$$H = \frac{p^2}{2m} + \frac{kx^2}{2}. \tag{6}$$

And the quantum version is

$$H = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2} = \frac{\hat{p}^2}{2m} + \frac{m(\omega\hat{x})^2}{2}, \tag{7}$$

which corresponds to a Schrödinger equation

$$\imath\frac{\partial}{\partial t}\Psi(x,t) = H\Psi = -\frac{\hbar^2}{2m}\Delta\Psi + \frac{m\omega^2x^2}{2}\Psi, \tag{8}$$

where $\omega = \sqrt{k/m}$.

If $H = \frac{\hat{p}^2}{2m} + \frac{m(\omega\hat{x})^2}{2}$ is treated as a scalar, we have

$$H = \hbar\omega aa^\dagger \tag{9}$$

where $a = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{\imath}{m\omega}\hat{p})$, $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{\imath}{m\omega}\hat{p})$. While in fact,

$$\hbar\omega a^\dagger a = \frac{m\omega^2}{2}(\hat{x}^2 - \frac{\hat{p}^2}{m^2\omega^2} + \frac{\imath}{m\omega}[\hat{x},\hat{p}]) = -\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} - \frac{\omega\hbar}{2}. \tag{10}$$

So the correct expression is

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}). \tag{11}$$

Without any proof, we claim that $H$ as an infinite dimensional Hermitian operator, has eigenvalues $\hbar\omega(n+\frac{1}{2})$ and associated eigenstates $|n\rangle$.

$$a|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{12}$$

## 1.3 From Quantum Harmonic Oscillator to the Transmon Qubit

Analogous to the spring model, let us start with the LC resonant circuit. The energy instead of oscillating between the kinetic energy of the mass and the potential enerby of the spring, in the system, the energy oscillates between the electrical energy (kinetic) in the capacitor $C$ and magnetic energy (potential) in the inductor $L$.

Take the flux as the time integral of the voltage, say

$$\Phi(t) = \int_{-\infty}^{t} V(t')\mathrm{d}t', \tag{13}$$

the energy for the capacitor and the inductor can be represented as

$$\mathcal{T}_C = \frac{1}{2}C\Phi_t^2, \quad \mathcal{U}_L = \frac{1}{2L}\Phi^2. \tag{14}$$

Before conducting further computation, we need to introduce several identities:

$$Q_t = I, \Phi_t = V, Q = CV, \Phi = LI. \tag{15}$$

And the Hamiltonian of the system is defined as

$$H = \frac{Q^2}{2C} + \frac{C(\frac{1}{\sqrt{LC}})^2\Phi^2}{2} = \frac{1}{2}CV^2 + \frac{1}{2}LI^2. \tag{16}$$

Compare the Equation 16 with Equation 7, we may take $m = C$, $\omega = \frac{1}{\sqrt{LC}}$, and we are getting the exact same form. To make it quantum mechanical, we can rewrite the $H$ in terms of $n := \frac{e^2}{2C}$ and $\phi = 2\pi\frac{\Phi}{\Phi_0}$, say

$$H = 4E_C n^2 + \frac{1}{2}E_L\phi^2, \tag{17}$$

where $E_C = \frac{e^2}{2C}$ and $E_L = \frac{\Phi_0^2}{4\pi^2 L}$, but it does not affect the form

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}). \tag{18}$$

Notice that for the system, the eigenvalues form a ladder where two adjacent energy levels differ $\hbar\omega$. This is not suitable for constructing a qubit since we only want to energy levels to serve as the $|0\rangle$ and $|1\rangle$. To introduce the nonlinearity required to modify the harmonic potential, we use the Josephson junction, which makes the energy levels no longer uniformly distribute. Replacing the linear inductor with a Josephson junction, the resulting Hamiltonian is

$$H = 4E_C n^2 - E_J \cos(\phi) \approx 4E_C n^2 + \frac{1}{2}E_L\phi^2 - \frac{1}{24}E_L\phi^4, \tag{19}$$

which is nonlinear in terms of $\phi$.

# 2 Single-qubit Gate in Qubit Control

Besides the original Hamiltonian of the system, we may insert some other energy, which results in

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) + \Omega V(t)(a - a^\dagger) = H_{LC} + H_d. \tag{20}$$

Truncate all higher order terms, say

$$a \approx |0\rangle\langle 1| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, a^\dagger \approx |1\rangle\langle 0| = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{21}$$

so

$$H \approx cI - \frac{\omega_q}{2}Z + V_d(t)\Omega Y, \tag{22}$$

where $\omega_q = (E_1 - E_0)/\hbar$.

Notice that the system will always rotating along the $z$-axis on the Bloch sphere, and we are interested in how the system behaves like relative to that. For a initial state $|\psi(0)\rangle$, its dynamics goes like

$$\imath\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{23}$$

Define $|\psi_{rf}(t)\rangle := e^{\imath H_{LC}t}|\psi(t)\rangle$, it satisfies

$$\begin{aligned}
&\imath\frac{\partial}{\partial t}|\psi_{rf}(t)\rangle = -H_{LC}e^{\imath H_{LC}t}|\psi(t)\rangle + e^{\imath H_{LC}t}H|\psi(t)\rangle \\
&= (-H_{LC} + e^{\imath H_{LC}t}He^{-\imath H_{LC}t})|\psi_{rf}(t)\rangle = e^{\imath H_{LC}t}H_d e^{-\imath H_{LC}t}|\psi_{rf}(t)\rangle \\
&= V_d(t)\Omega\begin{bmatrix} e^{\imath\lambda_0 t} & \\ & e^{\imath\lambda_1 t}\end{bmatrix} Y \begin{bmatrix} e^{-\imath\lambda_0 t} & \\ & e^{-\imath\lambda_1 t}\end{bmatrix} = V_d(t)\Omega\begin{bmatrix} & -\imath e^{\imath(\lambda_0-\lambda_1)t} \\ \imath e^{\imath(\lambda_1-\lambda_0)t} & \end{bmatrix} \\
&= V_d(t)\Omega(\cos(\delta t)Y - \sin(\delta t)X).
\end{aligned} \tag{24}$$

Here $\delta := \lambda_1 - \lambda_0$ is the spectral gap. Consider $V_d(t) = V_0 v(t) = V_0 s(t)\sin(\omega_d t + \phi)$, the Hamiltonian in the rotating frame becomes

$$\begin{aligned}
&V_0\Omega s(t)\Big[\sin(\omega_d t)\cos(\phi) + \cos(\omega_d t)\sin(\phi)\Big]\Big[\cos(\delta t)Y - \sin(\delta t)X\Big] \\
&= -\frac{1}{2}V_0\Omega s(t)(\cos(\phi)X + \sin(\phi)Y),
\end{aligned} \tag{25}$$

provided with $\omega_d = \delta$, and $\omega_d + \delta$ is big enough so that the fast rotating terms that will average to zero (rotating wave approximation). This indicates that we are able to rotate a qubit on the Bloch sphere.

# References


[1] David J Griffiths and Darrell F Schroeter. *Introduction to quantum mechanics*. Cambridge university press, 2018.

[2] Philip Krantz et al. "A quantum engineer's guide to superconducting qubits". In: *Applied physics reviews* 6.2 (2019).